3.2 Linear combination and linear independence of vectors

3.2.1 Linear combination of vectors

Geometric interpretation of vectors

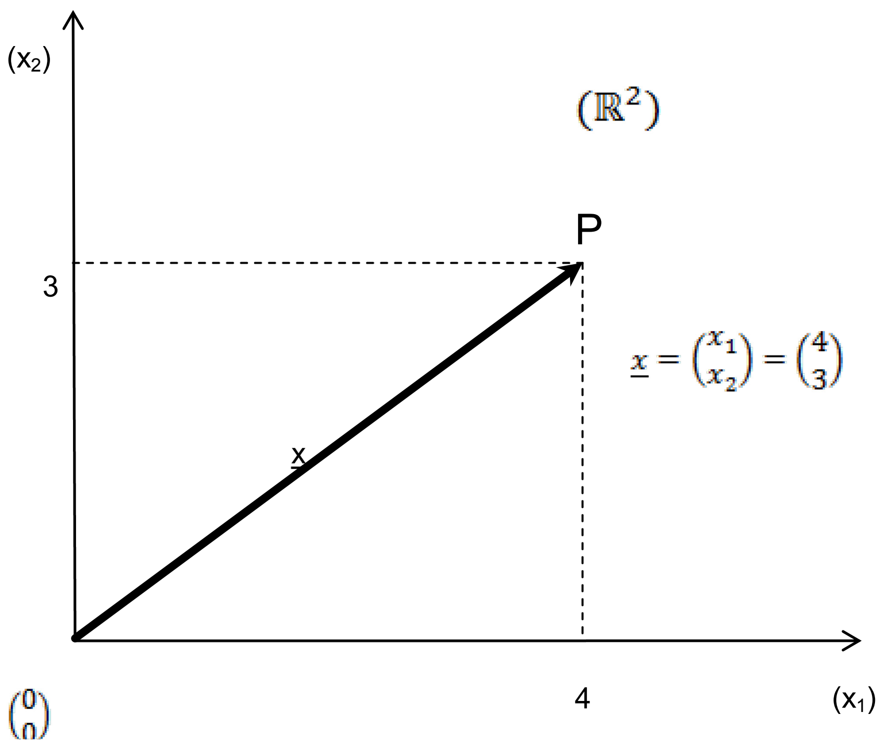
$\mathbf{R}^2$: Pairs of real numbers $\longleftrightarrow$ points in $\mathbf{x_1}$, $\mathbf{x_2}$ – **plane,**

e.g.. point $\begin{pmatrix} \mathbf{4} \\ \mathbf{3} \end{pmatrix}$

$\updownarrow$

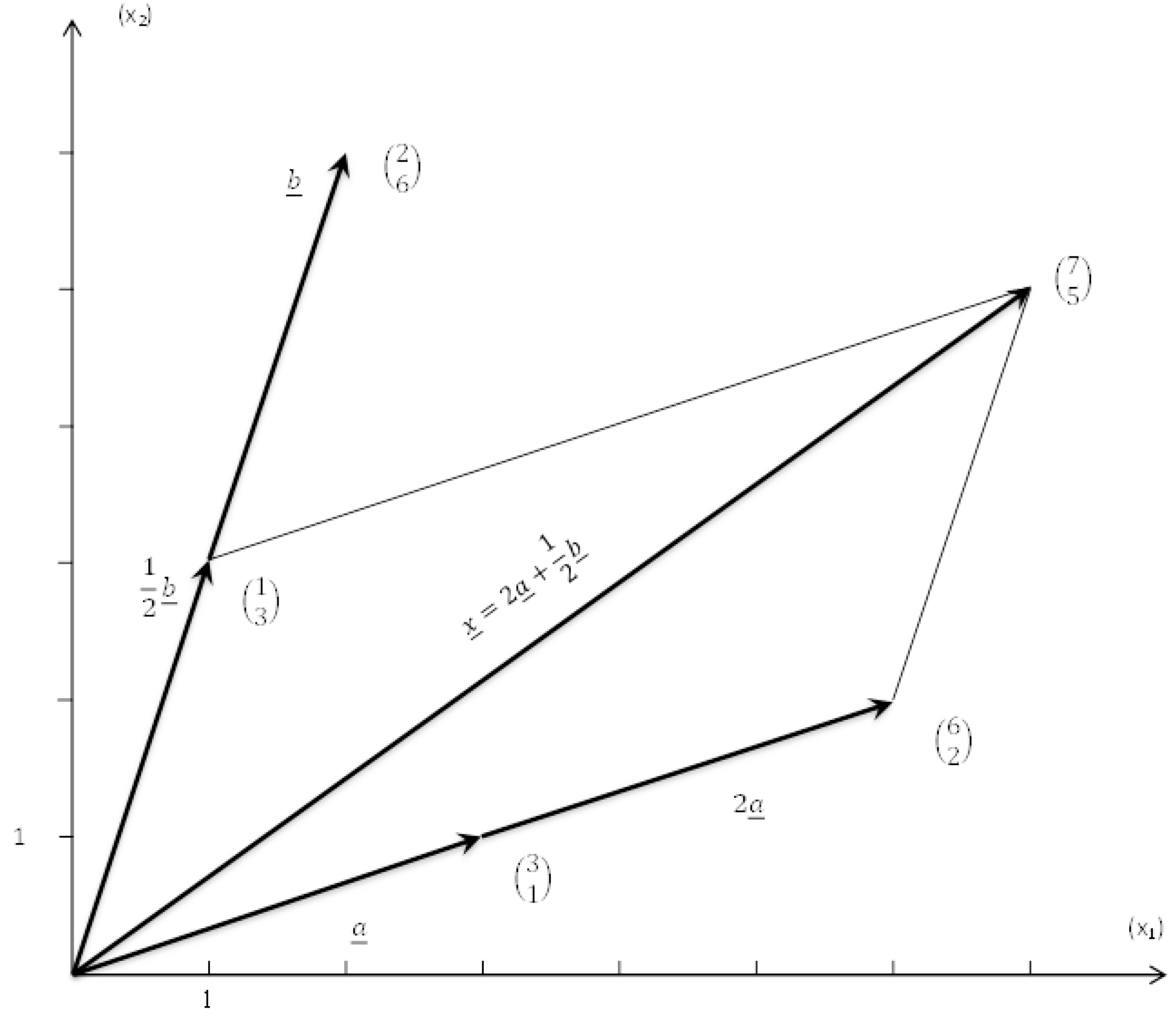
vector $\underline{\mathbf{x}} = \begin{pmatrix} \mathbf{4} \\ \mathbf{3} \end{pmatrix}$

directed distance (length, direction) also $\overrightarrow{\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{4} \\ \mathbf{3} \end{pmatrix}}$.



$$\underbrace{\alpha \cdot \underline{\mathbf{a}}; \quad \underline{\mathbf{a}}_1 + \underline{\mathbf{a}}_2}$$

combination: e.g. $2\underline{\mathbf{a}} + \frac{1}{2}\underline{\mathbf{b}}$



R^n : n-tuples of real numbers ↔ points in a n-dimensional space

↕

Vectors with the dimension n

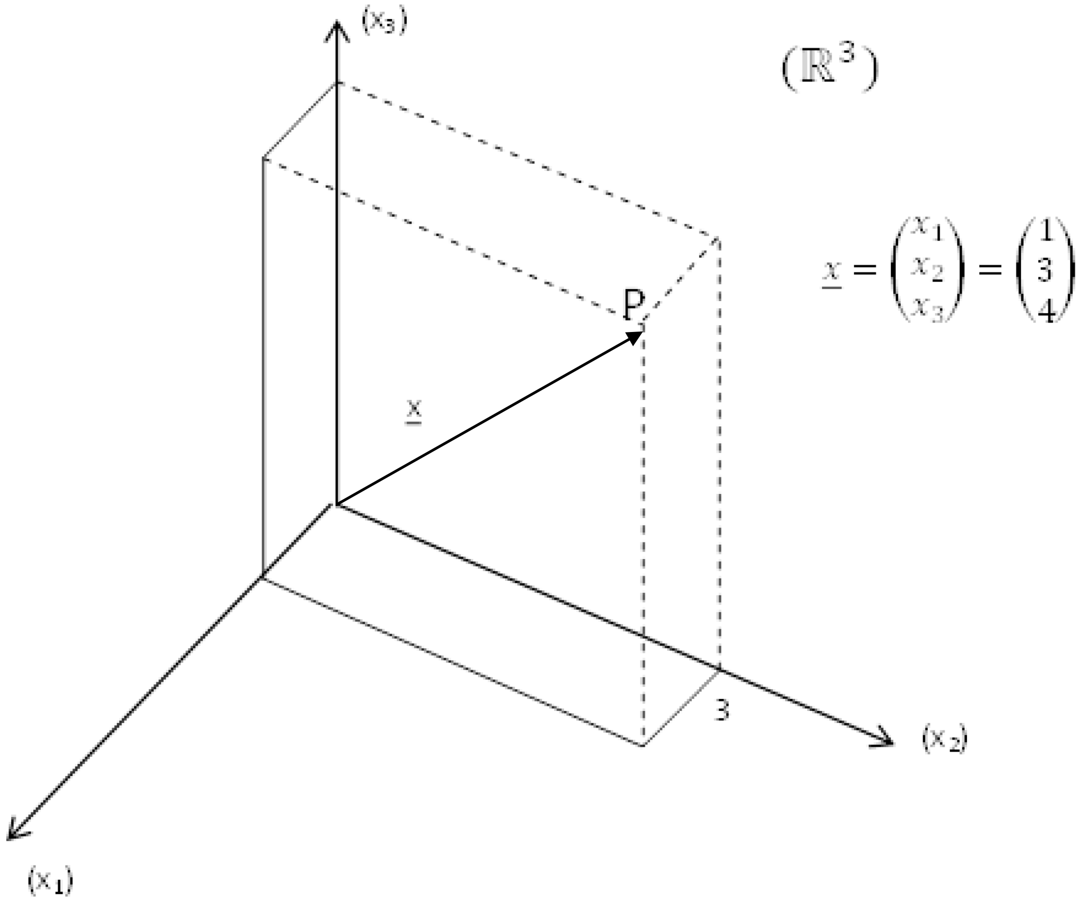
\+ structure:

- Multiplication with numbers
- addition
- additional characteristics

⇓

R^n : n-dimensional vector space

R^1 - line, R^2 - plane, R^3 - space



Definition:

If the vectors $\underline{\mathbf{a}}_1, \cdots, \underline{\mathbf{a}}_k \in \mathbf{R}^n$ and k real numbers $\alpha_1, \cdots, \alpha_k$ are given,

the vector $\underline{\mathbf{b}} = \sum_{i=1}^{k} \alpha_i \, \underline{\mathbf{a}}_i$ is a linear combination of the vectors $\underline{\mathbf{a}}_1, \cdots, \underline{\mathbf{a}}_k$

Examples:

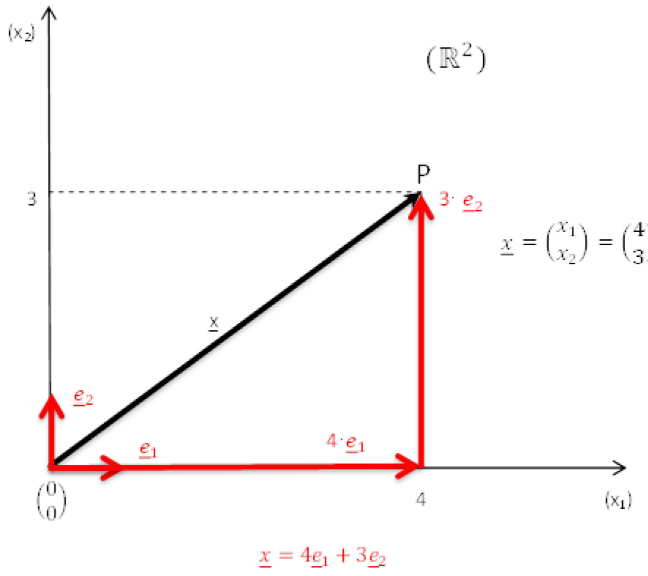
1) $\underline{\mathbf{x}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ – **linear combination of** $\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2$;

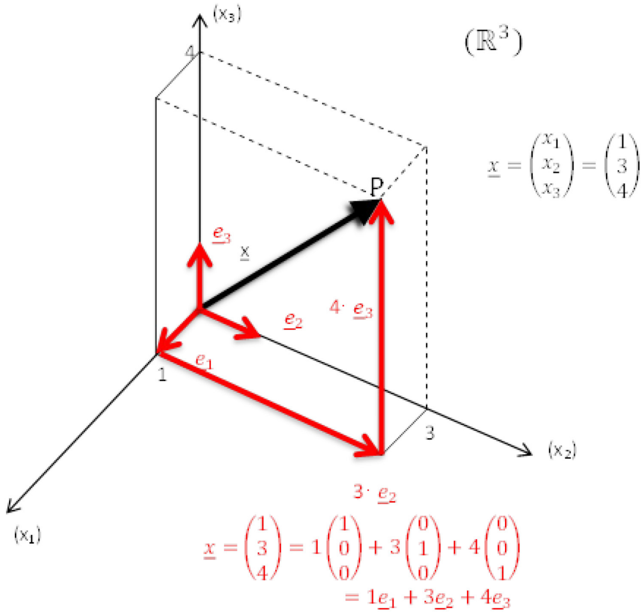
$\underline{\mathbf{x}} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ – **linear combination of** $\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3$;

$\underline{\mathbf{a}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \underline{\mathbf{b}} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ We have:

$\underline{\mathbf{x}} = 2\underline{\mathbf{a}} + 0{,}5\underline{\mathbf{b}}$ is a linear combination of $\underline{a}$ and $\underline{b}$ and

$\underline{\mathbf{x}} = 7\underline{\mathbf{e}}_1 + 5\underline{\mathbf{e}}_2$ is a linear combination of $\underline{e}_1$ and $\underline{e}_2$.





2)

Linear equation system (LES)

$$\begin{aligned} 2x_1 - x_2 + 0{,}5\,x_3 + x_4 &= 6{,}4 \\ x_1 \quad -2\,x_3 \quad &= 0 \\ -x_1 - 2\,x_2 + x_3 + 3\,x_4 &= 10 \end{aligned}$$

Equivalent way of writing with a vector:

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} x_1 + \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} x_2 + \begin{pmatrix} 0,5 \\ -2 \\ 1 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} x_4 = \begin{pmatrix} 6,4 \\ 0 \\ 10 \end{pmatrix}$$

$x_1 \in R$

$\underline{a}_1 \in R^3$ $\underline{a}_2 \in R^3$ $\underline{a}_3 \in R^3$ $\underline{a}_4 \in R^3$ $\underline{b} \in R^3$

The vector $\underline{b}$ is a linear combination of $\underline{a}_1, ..., \underline{a}_4$.

Way of writing as a matrix:

$$\begin{pmatrix} 2 & -1 & 0,5 & 1 \\ 1 & 0 & -2 & 0 \\ -1 & -2 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6,4 \\ 0 \\ 10 \end{pmatrix}$$

Coefficient matrix

$\underline{A}$

$$\underline{\mathbf{A}} \cdot \underline{\mathbf{x}} = \underline{\mathbf{b}}$$

3.2.2 Linear independence of vectors

We consider systems of vectors:

(1) $\underline{\mathbf{a}}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \underline{\mathbf{a}}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Does α exist with $\underline{\mathbf{a}}_1 = \alpha \cdot \underline{\mathbf{a}}_2$? $\rightarrow$ No!

(The vectors are on different lines)!

(2) $\underline{\mathbf{a}}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \underline{\mathbf{a}}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Is it possible to get a vector as linear combination of the other vectors?

$\rightarrow$ Yes!

$$\underline{\mathbf{a}}_3 = 1 \cdot \underline{\mathbf{a}}_1 + 1 \cdot \underline{\mathbf{a}}_2$$

$$\Updownarrow$$

$$\underline{\mathbf{a}}_1 + \underline{\mathbf{a}}_2 - 1 \cdot \underline{\mathbf{a}}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \underline{\mathbf{0}}$$

Definition:

The vectors $\mathbf{R}^n : \underline{\mathbf{a}}_1, \ldots, \underline{\mathbf{a}}_k \in \mathbf{R}^n$ are called **linear independent** if

$\underline{\mathbf{0}} = \alpha_1 \cdot \underline{\mathbf{a}}_1 + \alpha_2 \ \underline{\mathbf{a}}_2 + \ldots + \alpha_k \cdot \underline{\mathbf{a}}_k$ always implies that

$\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$,

otherwise they are called linear dependent.

Remarks:

1) The definition includes a condition which says that it is only possible to express the nullvector as linear combination of the vectors $\underline{\mathbf{a}}_1, ..., \underline{\mathbf{a}}_k$ in a trivial way.

2) $\underline{\mathbf{a}}_1, ..., \underline{\mathbf{a}}_k$ are linear dependent , if and only if there is at least one vector $\alpha_i, \quad i \in \{1, ..., k\} \quad with \quad \alpha_i \neq 0$,which can be expressed as linear combination of the other vectors.

3) A subset of linear independent vectors is also linear independent.

Example: (1), (2) (see above)

Let k > n, then it holds: k-vectors out of R^n (more than n vectors) are always linear dependent.

The unit vectors

$$\underline{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \underline{\mathbf{e}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \cdots, \underline{\mathbf{e}}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{i-th coordinate} \\ \\ \\ \\ \end{matrix}, \cdots, \quad \underline{\mathbf{e}}_n = \begin{pmatrix} 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{pmatrix}$$

of R^n are linear independent.